

# Stock Market Insider Trading in Continuous Time with Imperfect Dynamic Information \*

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## Abstract

This paper studies the equilibrium pricing of asset shares in the presence of *dynamic* private information. The market consists of a risk-neutral informed agent who observes the firm value, noise traders, and competitive market makers who set share prices using the total order flow as a noisy signal of the insider's information. I provide a characterization of all optimal strategies, and prove existence of both Markovian and non Markovian equilibria by deriving closed form solutions for the optimal order process of the informed trader and the optimal pricing rule of the market maker. The consideration of non Markovian equilibrium is relevant since the market maker might decide to re-weight past information after receiving a new signal. Also, I show that *a)* there is a unique Markovian equilibrium price process which allows the insider to trade undetected, and that *b)* the presence of an insider increases the market informational efficiency, in particular for times close to dividend payment.

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# 1 Introduction

Although financial markets with informational asymmetries have been widely discussed in the market microstructure literature (see [5] and [13] for a review), the characterization of the optimal trading strategy of an investor who possesses superior information has been, until lately, largely unaddressed by the mathematical finance literature.

In recent years, with the development of enlargement of filtrations theory (see [11]), models of so called insider trading have been gaining attention in mathematical finance as well (see e.g. [1], [4] and [9]). The salient assumptions of these models are that *i*) the informational advantage of the insider is a functional of the stock price process (e.g. the insider might know in advance the maximum value the stock price will achieve), and that *ii*) the insider does not affect the stock price dynamics. But in fact, since equilibrium stock prices should clear the market, and thus depend on the future random demand of market participants, assuming that the informational advantage of the insider is a functional of the price process implies that she either knows the future demand processes of *all market participants*, or she knows that the price will – exogenously – converge to a fundamental that is known to her. Since the assumption of an omniscient insider is unrealistic, one would have to assume the latter. Nevertheless, since the presence of an insider – by assumption in these models – does not affect the price process, this raises the question of what makes the price converge to its fundamental value without information being released to the market.

Thus, from the market microstructure point of view, these modeling assumptions translate into *i*) imposing strong efficiency of the markets even without an insider providing, through her trading, information to the market – that is, assuming a priori that the price will converge to the fundamental value – and that *ii*) the less informed agents are *not* fully rational, since they do not try to infer the insider’s private signal from market data (since there is no feedback from insider trading to equilibrium price).

Part of the mathematical finance literature has tried to address these shortcomings by considering the informational content of stock prices, and optimal information-based trading, in a rational expectations equilibrium framework (see e.g. [2], [7]). In these models – to preserve tractability –

the private information of the insider has been generally assumed to be *static*. For example, in [2] and in [7] the insider knows *ex ante* the final value of the firm, and in [6] she knows *ex ante* the time of default of the company issuing the asset. This literature has shown that *i*) the presence of an insider on the market does not necessarily lead to arbitrage (i.e. the value function of the insider is finite), and that *ii*) the presence of insiders might be considered beneficial to the market, in the sense that it leads to higher information efficiency of the equilibrium price process.

Nevertheless, the assumption of insider's perfect foresight is unrealistic, since the fundamental value of the firm should be connected to elements (like future cash-flows, productivity, sales etc.) that have intrinsically an aleatory component. That is, a more natural assumption would be that the fundamental value is in itself a stochastic process, and that the insider can observe it directly – or at least observe it in a less noisy way than the other agents on the market.

Thus, in this paper I relax the assumption of static insider information, and study the equilibrium trading and price processes, as well as market efficiency, in a setting with *dynamic* private information.

The model I consider in this paper is a generalization of the static information setting of [2]. An earlier attempt to generalize this framework to include dynamic information is in [3]. This latter paper considers a much smaller set of admissible trading strategies and pricing rules, and has much more stringent assumptions on the parameters, than the ones considered in my work. Moreover, it shows the existence of *one* possible Markovian equilibrium, while my work characterizes *all* optimal strategies and establishes that there is a unique *Markovian inconspicuous* equilibrium price process, i.e. an equilibrium price that allows the insider to trade undetected and depends only on the total order process. Moreover, I identify this Markovian equilibrium in closed form, and show that the presence of an insider increases the market informational efficiency for times close to dividend payment. Furthermore, I show that even when the market parameters do not satisfy the conditions for the existence of a Markovian equilibrium, there exists a *non Markovian* inconspicuous equilibrium which I also identify in closed form. Additionally, I give characterization of all optimal trading strategies for the equilibrium price process. I show, based on this characterization, that in the case of non Markovian price process it is optimal for the insider to reveal her private information

not only at the terminal time, but also at some predefined interim times – thus bringing the market to higher efficiency than in the case of Markovian price process.

The remainder of the paper is organized as follows. Section 2 presents the model and the assumptions. Existence of Markovian equilibrium, and uniqueness of the inconspicuous Markovian equilibrium price process, are proved in Section 3. Existence of equilibrium for more general pricing functionals is demonstrated in the Section 4. Section 5 concludes.

## 2 The Model Setup

Consider a stock issued by a company with fundamental value given by the process  $Z_t$ , defined on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , and satisfying

$$Z_t = v + \int_0^t \sigma_z(s) dB_s^1$$

where  $B_t^1$  is a standard Brownian motion on  $\mathcal{F}_t$ ,  $v$  is  $N(0, \sigma)$  independent of  $\mathcal{F}_t^{B^1}$  for any  $t$ , and  $\sigma_z(s)$  is a deterministic function.

Then, if the firm value is observable, the fair stock price should be a function of  $Z_t$  and  $t$ . However, the assumption of the company value being discernable by the whole market in continuous time is counterfactual, and it will be more realistic to assume that this information is revealed to the market only at given time intervals (such as dividend payments times or when balance sheets are publicized).

In this model I therefore assume, without loss of generality, that the time of the next information release is  $t = 1$ , and the market terminates after that.<sup>1</sup> Hence, in this setting the stock can be viewed as a European option on the firm value with maturity  $T = 1$  and payoff  $f(Z_1)$ . In addition to this risky asset, there is a riskless asset that yields an interest rate normalized to zero for simplicity of exposition. In what follows it is assumed that all random variables are defined on the same stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ .

The microstructure of the market, and the interaction of market participants, is modeled as a

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<sup>1</sup>This is without loss of generality, since the extension to multiple information release times is straightforward.

generalization of [2]. There are three types of agents: noisy/liquidity traders, an informed trader (insider), and competitive market makers, all of whom are risk neutral. The agents differ in their information sets, and objectives, as follows.

- *Noisy/liquidity traders* trade for liquidity reasons, and their total demand at time  $t$  is given by a standard Brownian motion  $B_t^2$  independent of  $B^1$  and  $v$ .
- *Market makers* observe only the total market order process  $Y_t = \theta_t + B_t^2$ , where  $\theta_t$  is the total order of the insider, i.e. their filtration is  $\mathcal{F}_t^M = \bar{\mathcal{F}}_t^Y$ . Since they are competitive and risk neutral, on the basis of the observed information they set the price as

$$P(Y_{[0,t]}, t) = P_t = \mathbb{E}[f(Z_1) | \mathcal{F}_t^M]. \quad (2.1)$$

As in [7], I assume that market makers set the price as a function of weighted total order process at time  $t$ , i.e. I consider pricing functionals  $P(Y_{[0,t]}, t)$  of the following form

$$P(Y_{[0,t]}, t) = H\left(\int_0^t w(s) dY_s, t\right).$$

where  $w(s)$  is some positive deterministic function.

- *The informed investor* observes the price process  $P_t = H(\int_0^t w(s) dY_s, t)$  and the true firm value  $Z_t$ , i.e. her filtration is given by  $\mathcal{F}_t^I = \bar{\mathcal{F}}_t^{Z,P}$ . Since she is risk-neutral, her objective is to maximize the expected final wealth, i.e.

$$\sup_{\theta \in \mathcal{A}(H,w)} \mathbb{E}[X_1^\theta] = \sup_{\theta \in \mathcal{A}(H,w)} \mathbb{E}\left[(f(Z_1) - P_1)\theta_1 + \int_0^1 \theta_s - dP_s\right] \quad (2.2)$$

where  $\mathcal{A}(H,w)$  is the set of admissible trading strategies for the given price functional  $H\left(\int_0^t w(s) dY_s, t\right)$  which will be defined later. That is, the insider maximizes the expected value of her final wealth  $X_1^\theta$ , where the first term on the right hand side of equation (2.2) is the contribution to the final wealth due to a potential differential between price and fundamental at the time of information release, and the second term is the contribution to final wealth

coming from the trading activity.

Note that setting  $\sigma_z \equiv 0$ , the resulting market would be the static information one considered by [2].

Note also that the above market structure implies that the insider's optimal trading strategy takes into account the *feedback effect* i.e. the that prices react to her trading strategy according to equation (2.1). Identifying the optimal insider's strategy is equivalent to the problem of finding the rational expectations equilibrium of this market, i.e. a pair consisting of an *admissible* price functional and an *admissible* trading strategy such that: *a)* given the price functional the trading strategy is optimal, and *b)* given the trading strategy the price functional satisfies (2.1). To formalize this definition, we first need to define the sets of admissible pricing rules and trading strategies.

Although it is standard in the insider trading literature to limit the set of admissible strategies to absolutely continuous ones, in what follows I consider a much broader class of strategies given by the set of semimartingales satisfying some standard technical conditions that eliminate doubling strategies. The formal definition of the set of admissible trading strategies is summarized in the following definition.

**Definition 2.1** *An insider's trading strategy,  $\theta_t$ , is admissible for a given pricing rule  $(H(y, t), w(t))$  ( $\theta \in \mathcal{A}(H, w)$ ) if  $\theta_t$  is  $\mathcal{F}_t^I$  adapted semimartingale, and no doubling strategies are allowed i.e.*

$$\mathbb{E} \left[ \int_0^1 H^2 \left( \int_0^t w(s) d\theta_{s-} + \int_0^t w(s) dB_s^2, t \right) dt \right] < \infty. \quad (2.3)$$

Moreover, we call the insider's trading strategy *inconspicuous* if  $Y_t = \theta_t + B_t^2$  is a Brownian motion on its own filtration  $\mathcal{F}_t^Y$  (since in this case the presence of the insider is undetectable).

**Remark 2.1** *An equilibrium in which the optimal insider's trading strategy is inconspicuous is a desirable feature of any insider trading model, and I will show that in this setting such an equilibrium exists. In fact, given the potentially high cost associated with being identified as an insider, it might be reasonable to consider only this type of equilibrium.*

The definition of admissible pricing rules is a generalization of the one in [2]<sup>2</sup> with additional regularity condition 5 below which insures that, given the market maker's filtration, the total order process has finite variance. This generalization allows the market maker to re-weight her past information.

**Definition 2.2** *A pair of measurable functions,  $H \in C^{2,1}(\mathbb{R} \times [0, 1])$ ,  $H : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$  and  $w : [0, 1] \rightarrow \mathbb{R}_+ \setminus \{0\}$ , is an admissible pricing rule  $((H, w) \in \mathcal{H})$  if and only if:*

1. *The weighting function,  $w(t)$ , is a piecewise positive constant function given by*

$$w(t) = \sum_{i=1}^n \sigma_y^i 1_{\{t \in (t_{i-1}, t_i]\}} \quad (2.4)$$

*where  $0 = t_0 < t_1 < \dots < t_n = 1$  and  $\sum_{i=1}^n (\sigma_y^i)^2 = 1$ .*

*This condition doesn't cause loss of generality because: a) it was shown by [7], in the static private information case, that in the equilibrium  $w'(t) = 0$  and b) it is always possible to re-scale  $w$  to have  $\sum_{i=1}^n (\sigma_y^i)^2 = 1$ .*

2.  $\mathbb{E} \left[ \int_0^1 H^2 \left( \int_0^t w(s) dB_s^2, t \right) dt \right] < \infty.$

3.  $\mathbb{E} \left[ H^2 \left( \int_0^1 w(s) dB_s^2, 1 \right) \right] < \infty.$

*The two conditions above, together with equation(2.3), rule out doubling strategies.*

4.  *$y \rightarrow H(y, t)$  is increasing for each fixed  $t$ , that is the price increases if the stock demand increases.*

5.  $\mathbb{E} \left[ \left( h_i^{-1} \left( \mathbb{E} [f(Z_1) | \mathcal{F}_{t_i}^Z] \right) \right)^2 \right] < \infty$  *where  $h_i^{-1}$  is the inverse of  $H(y, t_i)$ .*

*Moreover,  $H$  is a rational pricing rule if, for a given  $\theta$ , it satisfies*

$$H \left( \int_0^t w(s) dY_s, t \right) = \mathbb{E} [f(Z_1) | \mathcal{F}_t^M].$$

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<sup>2</sup>Setting  $w(s) \equiv 1$  will make conditions 2-4 exactly the same as in [2]

**Remark 2.2** Due to condition 4 on the admissible pricing rules, the insider can infer the total order process from the price process by inverting  $H(\int_0^t w(s)dY_s, t) = P_t$ . Therefore, since I will be considering rational expectations equilibria, and because  $w(s)$  is strictly positive, she can infer the total order process  $Y_t$  and, since she knows her own total order process  $\theta_t$ , she can deduce  $B_t^2 = Y_t - \theta_t$  from it. As a consequence, the filtration of the insider can be written as  $\mathcal{F}_t^I = \mathcal{F}_t^{B^2, Z} = \mathcal{F}_t^{B^2, B^1} \vee \sigma(v)$ , where  $\sigma(v)$  is the sigma algebra generated by the random variable  $v$ .

Given these definitions of admissible pricing rules and trading strategies, it is now possible to formally define the market equilibrium as follows.

**Definition 2.3** A pair  $((H^*, w^*), \theta^*)$  is an equilibrium if  $(H^*, w^*)$  is an admissible pricing rule,  $\theta^*$  is admissible strategy, and:

1. Given  $\theta^*$ ,  $(H^*, w^*)$  is a rational pricing rule, i.e. it satisfies

$$H\left(\int_0^t w(s)dY_s, t\right) = \mathbb{E}[f(Z_1)|\mathcal{F}_t^M].$$

2. Given  $(H^*, w^*)$ ,  $\theta^*$  solves the optimization problem

$$\sup_{\theta \in \mathcal{A}(H^*, w^*)} \mathbb{E}\left[(f(Z_1) - P_1)\theta_1 + \int_0^1 \theta_s - dP_s\right]$$

Moreover, a pricing rule  $(H^*(y, t), w^*(t))$  is an inconspicuous equilibrium pricing rule if there exists an inconspicuous insider trading strategy  $\theta^*$  such that  $((H^*, w^*), \theta^*)$  is an equilibrium.

Additionally, to define a well behaved problem I impose the following technical conditions on the model parameters.

**Assumption 2.1** The fundamental value of the risky stock,  $F(z, t)$ , given by

$$F(Z_t, t) = \mathbb{E}[f(Z_1)|\mathcal{F}_t^Z] \tag{2.5}$$



is well defined and is a square integrable martingale, i.e.

$$\mathbb{E} [f^2(Z_1)] < \infty, \quad (2.6)$$

and  $f(\cdot)$  is an increasing function.

**Assumption 2.2** The variance of the firm value,  $\Sigma_z(t) = \int_0^t \sigma_z^2(s) ds$ , is finite for any  $t$ .

**Remark 2.3** Since the final payoff of the stock is given by  $f(Z_1)$ , the above assumption implies that it is always possible to redefine the function  $f$  so that

$$\sigma^2 = 1 - \Sigma_z(1). \quad (2.7)$$

In what follows, I will always assume that this equality holds.

### 3 The Markovian Equilibrium

In this section I address the problem of existence and uniqueness of an equilibrium given by Definition 2.3 in the case of Markovian pricing rule i.e. I consider  $w(t) \equiv 1$ . Before stating the main result of this section, I need to impose additional conditions on the model to insure that the problem is well-posed.

**Assumption 3.1** For any  $t \in [0, 1)$  we have

$$\int_0^t (\Sigma_z(s) + \sigma^2 - s)^{-2} ds < \infty \quad (3.1)$$

and either

$$\int_0^1 (\Sigma_z(s) + \sigma^2 - s)^{-2} ds < \infty \quad (3.2)$$

or

$$\lim_{t \rightarrow 1} \int_0^t \frac{1}{|\Sigma_z(s) + \sigma^2 - s|} ds = \infty \quad (3.3)$$

The above assumption is needed for the filtering problem of the market maker to be well defined.

**Assumption 3.2** *There exists a  $t^* \in [0, 1)$  such that*

$$1 - t > \int_t^1 \sigma_z^2(s) ds \quad (3.4)$$

*for any  $t \geq t^*$  and  $\sigma_z(t)$  is continuous on  $[t^*, 1]$ .*

*Moreover, for all  $t \in [0, 1]$  we have*

$$\Sigma_z(t) - t + \sigma^2 \geq 0. \quad (3.5)$$

This assumption insures that: a) close to the market terminal time, the insider's signal is more precise than the market maker's (i.e.  $\mathbb{E} \left[ (Z_1 - \mathbb{E} [Z_1 | \mathcal{F}_t^M])^2 | \mathcal{F}_t^M \right] > \mathbb{E} \left[ (Z_1 - \mathbb{E} [Z_1 | \mathcal{F}_t^I])^2 | \mathcal{F}_t^I \right]$ ), and b) that the insider's signal is always at least as precise as the market maker's.

**Remark 3.1** *Notice that Assumptions 2.2, 3.1 and 3.2 guarantee that when condition (3.2) is not satisfied*

$$\lambda(t) = \exp \left\{ - \int_0^t \frac{1}{\Sigma_z(s) + \sigma^2 - s} ds \right\} \xrightarrow{t \rightarrow 1} 0,$$

*and that if  $\Xi(t) = \int_0^t \frac{1 + \sigma_z^2(s)}{\lambda^2(s)} ds \xrightarrow{t \rightarrow 1} \infty$ , then*

$$\lim_{t \rightarrow 1} \lambda^2(t) \Xi(t) \log \log (\Xi(t)) = 0. \quad (3.6)$$

*Furthermore, assumption 3.2 can be relaxed by replacing (3.4) with condition (3.6).*

PROOF. See Appendix A

Now we are in the position to state the main result of this section which is summarized in the next theorem.

**Theorem 3.1** *Suppose that Assumptions 2.1, 2.2, 3.1 and 3.2 are satisfied. Then the pair  $(H^*, \theta^*)$ ,*

where  $H^*$  satisfies

$$H_t(y, t) + \frac{1}{2}H_{yy}(y, t) = 0 \quad (3.7)$$

$$H(y, 1) = f(y), \quad (3.8)$$

i.e.  $H^*(y, t) = \mathbb{E} [f(y + B_1^2 - B_t^2)]$  and

$$\theta_t^* = \int_0^t \frac{Z_s - Y_s}{\Sigma_z(s) - s + \sigma^2} ds, \quad (3.9)$$

is an equilibrium. Moreover, the pricing rule  $H^*$  is the unique inconspicuous equilibrium pricing rule in  $\mathcal{H}$ . Furthermore, given this pricing rule  $H^*$ , the trading strategy  $\theta^*$  is optimal in  $\mathcal{A}(H^*)$  for the insider if and only if

1. The process  $\theta_t^*$  is continuous and has bounded variation.
2. The total order,  $Y_t^* = \theta_t^* + B_t^2$ , satisfies  $Y_1^* = Z_1$ .

Therefore, when the parameters of the market satisfy the stated assumptions, there exists a unique Markovian pricing rule such that: *a)* at least one optimal trading strategy of the insider, given by (3.9), is *increasing* market efficiency during *all* trading periods since the insider pushes the price to the fundamental value of the stock, *b)* due to 1, the variance of the risky asset is not influenced by insider's trading if she trades optimally, and *c)* the insider presence increases market efficiency close to the market termination time due to 2.

I will prove this theorem in three propositions that focus on different aspects of the equilibrium. In particular, the propositions will address: *a)* characterization of the optimal insider trading strategy, *b)* existence of the equilibrium, and *c)* uniqueness of the inconspicuous pricing rule.

The conclusion of Theorem 3.1 is driven by the following result: for any pricing rule in  $\mathcal{H}$  satisfying equation (3.7), there exists a finite upper bound on the informed agent's value function which is attained by a trading strategy which is not detectable by the market maker, not locally correlated with noisy trades, and such that all the private information is revealed only at time  $t = 1$ .

Thus, this result gives the characterization of the optimal insider's trading strategy in a slightly more general form than stated in Theorem 3.1. This is summarized in the following proposition.

**Proposition 3.1** *Suppose that Assumptions 2.1, 2.2, 3.1 and 3.2 are satisfied. Then, given an admissible pricing rule  $H \in \mathcal{H}$  satisfying the partial differential equation (PDE) (3.7), an admissible trading strategy  $\theta^* \in \mathcal{A}(H)$  is optimal for the insider if and only if:*

1. *The process  $\theta_t^*$  is continuous and has bounded variation.*
2. *The total order,  $Y_t^* = \theta_t^* + B_t^2$ , satisfies*

$$h(Y_1^*) = H(Y_1^*, 1) = f(Z_1), \quad (3.10)$$

where  $h(y) = H(y, 1)$  and  $f(Z_1)$  is the final payoff of the asset.

PROOF.

(Sufficiency) For any admissible trading strategy, by using integration by parts for semimartingales ([14], Corollary II.6.2, p. 68), we have

$$\mathbb{E}[X_1^\theta] = \mathbb{E}\left[\int_0^1 (F(Z_s, s) - H(Y_{s-}, s))d\theta_s + \int_0^1 \theta_{s-}dF(Z_s, s) + [\theta, F(Z, \cdot) - H(Y, \cdot)]_1\right].$$

By applying Itô formula for semimartingales ([14], Theorem II.6.33, p. 81) to  $H(y, t)$  and  $F(z, t)$ , and using the fact that  $F(Z_t, t)$  is a true martingale, we get

$$\begin{aligned} F(Z_t, t) &= F(z_0, 0) + \int_0^t F_z(Z_s, s)dZ_s \\ H(Y_t, t) &= H(0, 0) + \int_0^t H_y(Y_{s-}, s)dY_s + \int_0^t H_t(Y_{s-}, s)ds \\ &\quad + \frac{1}{2} \int_0^t H_{yy}(Y_{s-}, s)d[Y]_s + \sum_{s \leq t} [\Delta H(Y_s, s) - H_y(Y_{s-}, s)\Delta Y_s]. \end{aligned}$$

Since  $Y_t = \theta_t + B_t^2$ , we have that  $[Y]_t = t + \langle \theta^c \rangle_t + 2 \langle \theta^c, B^2 \rangle_t + \sum_{s \leq t} (\Delta \theta_s)^2$ . Therefore, using

the fact that  $H(y, t)$  satisfies equation (3.7), we have that

$$\begin{aligned} H(Y_t, t) &= H(0, 0) + \int_0^t H_y(Y_{s-}, s) dY_s^c + \frac{1}{2} \int_0^t H_{yy}(Y_{s-}, s) d\langle \theta^c \rangle_s \\ &+ \int_0^t H_{yy}(Y_{s-}, s) d\langle \theta^c, B^2 \rangle_s + \sum_{s \leq t} \Delta H(Y_s, s). \end{aligned}$$

Therefore, by Theorem 26.6 of [8], and Theorem II.6.29 of [14], we have (notice that  $Z_s$  and  $B_t^2$  are continuous)

$$\begin{aligned} [\theta, F(Z, \cdot)]_1 &= \int_0^1 F_z(Z_s, s) d[\theta^c, Z]_s \\ [\theta, H(Y, \cdot)]_1 &= \int_0^1 H_y(Y_{s-}, s) d[\theta^c]_s + \int_0^1 H_y(Y_{s-}, s) d[\theta^c, B^2]_s + \sum_{s \leq 1} \Delta H(Y_s, s) \Delta \theta_s. \end{aligned}$$

On the other hand, consider a function

$$J(y, z) = \int_y^{y^*(z)} (f(z) - H(x, 1)) dx,$$

where  $y^*(z)$  is the solution of  $H(y^*(z), 1) = f(z)$ . Let

$$V(y, z, t) = \mathbb{E} \left[ J \left( y + B_1^2 - B_t^2, z + \int_t^1 \sigma_z(s) dB_s^1 \right) \right]. \quad (3.11)$$

This function is well defined (it is easy to check that  $\mathbb{E} [|J(B_1^2, Z_1)|] < \infty$ ) and satisfies the partial differential equation

$$V_t(y, z, t) + \frac{1}{2} V_{yy}(y, z, t) + \frac{\sigma_z^2(t)}{2} V_{zz}(y, z, t) = 0 \quad (3.12)$$

with terminal condition  $V(y, z, 1) = J(y, z)$ . Therefore  $V(y, z, 1) \geq V(y^*(z), z, 1) = 0$  for any fixed  $z$  and any  $y \neq y^*(z)$ . Moreover, since  $H(y, 1)$  is a nondecreasing continuous function of

$y$ , we can use the monotone convergence theorem to obtain

$$\begin{aligned}
\lim_{\Delta \rightarrow 0+} \frac{V(y + \Delta, z, t) - V(y, z, t)}{\Delta} &= \lim_{\Delta \rightarrow 0+} \frac{\mathbb{E} \left[ \int_{y+\Delta+B_1^2-B_t^2}^{y+B_1^2-B_t^2} \left( f(z + \int_t^1 \sigma_z(s) dB_s^1) - H(x, 1) \right) dx \right]}{\Delta} \\
&= -F(z, t) - \mathbb{E} \left[ \lim_{\Delta \rightarrow 0+} \frac{\int_{y+\Delta+B_1^2-B_t^2}^{y+B_1^2-B_t^2} H(x, 1) dx}{\Delta} \right] \\
&= \mathbb{E} [H(y + B_1^2 - B_t^2, 1)] - F(z, t).
\end{aligned}$$

Thus, due to the definition of an admissible pricing rule, we have

$$\lim_{\Delta \rightarrow 0+} \frac{V(y + \Delta, z, t) - V(y, z, t)}{\Delta} + F(z, t) - H(y, t) = 0. \quad (3.13)$$

The same argument can be applied to the left derivative of  $V$  with respect to  $y$  to obtain

$$V_y(y, z, t) + F(z, t) - H(y, t) = 0. \quad (3.14)$$

As a consequence, we can express  $\mathbb{E} [X_1^\theta]$  in terms of  $V$  as (notice that  $\int_0^t B_t^2 dF(Z_t, t)$  is a martingale)

$$\begin{aligned}
\mathbb{E} [X_1^\theta] &= \mathbb{E} \left[ - \int_0^1 V_y(Y_{s-}, Z_s, s) d\theta_s - \int_0^1 V_z(Y_{s-}, Z_s, s) dZ_s - \int_0^1 V_{zy}(Y_{s-}, Z_s, s) d[\theta^c, Z]_s \right. \\
&\quad \left. - \int_0^1 V_{yy}(Y_{s-}, Z_s, s) d[\theta^c]_s - \int_0^1 V_{yy}(Y_{s-}, Z_s, s) d[\theta^c, B^2]_s - \sum_{s \leq 1} \Delta V_y(Y_s, Z_s, s) \Delta \theta_s \right].
\end{aligned}$$

On the other hand, by applying the Itô formula for semimartingales to  $V$  directly ([14], Theorem II.6.33, p. 81) we get

$$\begin{aligned}
\mathbb{E} [V(Y_1, Z_1, 1)] &= \mathbb{E} \left[ V(0, Z_0, 0) - X_1^\theta + \int_0^1 V_y(Y_{s-}, Z_s, s) dB_s^2 \right. \\
&\quad \left. - \frac{1}{2} \int_0^1 V_{yy}(Y_{s-}, Z_s, s) d[\theta^c]_s + \sum_{s \leq 1} [\Delta V(Y_s, Z_s, s) - V_y(Y_s, Z_s, s) \Delta Y_s] \right].
\end{aligned}$$

Notice that, due to the definition of the fundamental value  $F$  and of admissible pricing rule

$H$ , we have  $\mathbb{E} \left[ \int_0^1 V_z(Y_s, Z_s, s) dB_s^2 \right] = 0$ . Therefore

$$\begin{aligned} \mathbb{E} \left[ X_1^\theta \right] &= \mathbb{E} \left[ V(0, Z_0, 0) - V(Y_1, Z_1, 1) - \frac{1}{2} \int_0^1 V_{yy}(Y_{s-}, Z_s, s) d[\theta^c]_s \right. \\ &\quad \left. + \sum_{s \leq 1} [\Delta V(Y_s, Z_s, s) - V_y(Y_s, Z_s, s) \Delta Y_s] \right]. \end{aligned}$$

Moreover, due to the properties of  $V$  we have

$$\sum_{s \leq 1} (\Delta V(Y_s, Z_s, s) - V_y(Y_s, Z_s, s) \Delta Y_s) \leq 0, \quad (3.15)$$

$$- \int_0^1 \frac{V_{yy}(Y_s, Z_s, s)}{2} d[\theta^c]_s \leq 0, \quad (3.16)$$

$$-V(1, Y_1, Z_1) \leq -V(1, y^*(Z_1), Z_1). \quad (3.17)$$

The above inequalities become equalities if and only if the following conditions hold:  $\Delta\theta = 0$  for equation (3.15);  $[\theta^c]_1 = 0$  for equation (3.16);  $H(Y_1^*, 1) = f(Z_1)$  for equation (3.17).

Therefore, for any function  $V$  satisfying equations (3.12), (3.14) and the final condition given by  $V(y, z, 1) \geq V(y^*(z), z, 1) = 0$  for every  $z$  and any  $y \neq y^*(z)$  (where  $y^*(z)$  is the solution of  $H(y^*(z), 1) = f(z)$ ), we have that

$$\mathbb{E} \left[ X_1^\theta \right] \leq V(0, Z_0, 0).$$

This expression holds with equality if and only if  $\theta$  is continuous and condition (3.10) is satisfied. Hence, if  $\theta$  is such that these conditions are satisfied, then it is optimal.

(Necessity) Consider the continuous martingale given by

$$X_t = G(Z_t, t) = \mathbb{E} \left[ h^{-1}(f(Z_1)) | \mathcal{F}_t^I \right].$$

This martingale is well defined since  $H$  is an admissible pricing rule.

Consider  $\theta_t = \int_0^t \frac{X_s - Y_s}{1-s} ds$ . In this case, we can solve the stochastic differential equation for

$Y$  to get

$$Y_t = X_t - (1-t) \left( v + \int_0^t \frac{1}{1-s} dX_s - \int_0^t \frac{1}{1-s} dB_s^2 \right).$$

Notice that  $Y_t$  is continuous, therefore  $\theta_t$  has bounded variation almost surely. Moreover,  $H(Y_1, 1) = f(Z_1)$  almost surely, hence this choice of  $\theta$  gives

$$\mathbb{E} \left[ X_1^\theta \right] = V(0, Z_0, 0).$$

Since, by the sufficiency proof, we have that for any  $\tilde{\theta}_t$  which is either not continuous or does not satisfy equation (3.10)

$$\mathbb{E} \left[ X_1^{\tilde{\theta}} \right] < V(0, Z_0, 0) = \mathbb{E} \left[ X_1^\theta \right],$$

we know that any such  $\tilde{\theta}_t$  is not optimal. ■

From this characterization result, it follows that the  $\theta_t^*$  given by (3.9) is an optimal insider trading strategy given an admissible pricing rule  $H^*$  satisfying (3.7) and (3.8). Establishing the rationality of the pricing rule  $H^*$ , on the other hand, is not so direct. Therefore, to set up the stage for proving that the  $(H^*, \theta^*)$  given in Theorem 3.1 is indeed an equilibrium, we first need to demonstrate the following lemma.

**Lemma 3.1** *Consider the process  $Y_t$  satisfying the stochastic differential equation*

$$dY_s = \frac{Z_s - Y_s}{\Sigma_z(s) - s + \sigma^2} ds + dB_s^2,$$

with

$$Z_t = v + \int_0^t \sigma_z(s) dB_s^1,$$

where  $B_t^1$  and  $B_t^2$  are two independent standard Brownian motions,  $v$  is  $N(0, \sigma)$  independent of  $\mathcal{F}_1^{B^1, B^2}$  and  $\Sigma_z(t) = \int_0^t \sigma_z^2(s) ds$ . Suppose that  $\sigma$ ,  $\sigma_z(t)$  and  $\Sigma_z(t)$  satisfy Assumptions 2.1, 2.2, 3.1 and 3.2. Then, on the filtration  $\bar{\mathcal{F}}_t^Y$ , the process  $Y_t$  is a standard Brownian motion and  $Y_1 = Z_1$ .



PROOF. Fix any  $T \in [0, 1)$ . From Theorem 10.3 of [12] (note that, due to Assumption 3.1, the conditions of the theorem are satisfied), we have that on the filtration  $(\mathcal{F}_t^Y)_{t \leq T}$  the stochastic differential equation for  $Y$  is

$$dY_s = \frac{m_s - Y_s}{\Sigma_z(s) - s + \sigma^2} ds + dB_s^Y,$$

with

$$dm_s = \frac{\gamma_s}{\Sigma_z(s) - s + \sigma^2} dB_s^Y,$$

where  $B_t^Y$  is Brownian motion on  $\mathcal{F}_t^Y$ , and  $\gamma_s$  satisfies the following ordinary differential equation (ODE)

$$\dot{\gamma}_s = \sigma_z^2(s) - \frac{\gamma_s^2}{(\Sigma_z(s) - s + \sigma^2)^2}$$

with initial condition  $\gamma_0 = \sigma^2$ .

Notice that  $\gamma_s = \Sigma_z(s) - s + \sigma^2$  is the unique solution of this ODE and initial condition. Therefore on  $(\mathcal{F}_t^Y)_{t \leq T}$ , the process  $Y$  satisfies

$$dY_s = \frac{B_s^Y - Y_s}{\Sigma_z(s) - s + \sigma^2} ds + dB_s^Y.$$

The unique strong solution of this stochastic differential equation on  $[0, T]$  is  $Y_s = B_s^Y$  (see [10], Example 5.2.4). Hence, on the interval  $[0, 1)$ , the process  $Y$  is a Brownian motion on its own (completed) filtration. By continuity of  $Y$ , this process is a Brownian motion on  $[0, 1]$ . To prove that  $Y_1 = Z_1$ , notice that

$$Y_t^* = Z_t + \lambda(t) \left( -v + \int_0^t \frac{1}{\lambda(s)} dB_s^2 - \int_0^t \frac{\sigma_z(s)}{\lambda(s)} dB_s^1 \right)$$

where  $\lambda(t) = \exp \left\{ - \int_0^t \frac{1}{\Sigma_z(s) + \sigma^2 - s} ds \right\}$ .

Note that a random variable  $\int_0^t \frac{1}{\lambda(s)} dB_s^2 - \int_0^t \frac{\sigma_z(s)}{\lambda(s)} dB_s^1$  is normally distributed with mean 0 and variance  $\int_0^t \frac{1 + \sigma_z^2(s)}{\lambda^2(s)} ds$ . Therefore, due to the Assumption 3.1 (and in particular condition (3.3)), if  $\lim_{t \rightarrow 1} \int_0^t \frac{1 + \sigma_z^2(s)}{\lambda^2(s)} ds < \infty$ , then  $Y_1 = Z_1$ .

On the other hand, if  $\lim_{t \rightarrow 1} \int_0^t \frac{1+\sigma_z^2(s)}{\lambda^2(s)} ds = \infty$ , consider the process

$$X_t = \int_0^t \frac{1}{\lambda(s)} dB_s^2 - \int_0^t \frac{\sigma_z(s)}{\lambda(s)} dB_s^1,$$

and a change of time  $\tau(t)$  given by

$$\int_0^{\tau(t)} \frac{1+\sigma_z^2(s)}{\lambda^2(s)} ds = t.$$

Then,  $W_s = X_{\tau(s)}$  is a Brownian motion. Hence, we can use the law of iterated logarithm to get

$$\begin{aligned} \limsup_{s \rightarrow \infty} \frac{W_s}{\sqrt{2s \log \log s}} &= 1 \\ \liminf_{s \rightarrow \infty} \frac{W_s}{\sqrt{2s \log \log s}} &= -1 \end{aligned}$$

or, in the original time,

$$\begin{aligned} \limsup_{t \rightarrow 1} \frac{X_t}{\sqrt{2\Xi(t) \log \log(\Xi(t))}} &= 1 \\ \liminf_{t \rightarrow 1} \frac{X_t}{\sqrt{2\Xi(t) \log \log(\Xi(t))}} &= -1 \end{aligned}$$

where  $\Xi(t) = \int_0^t \frac{1+\sigma_z^2(s)}{\lambda^2(s)} ds$ . Since, due to the Assumptions 2.2, 3.1 and 3.2, we have

$$\lim_{t \rightarrow 1} \lambda^2(t) \Xi(t) \log \log(\Xi(t)) = 0,$$

it follows that  $\lim_{t \rightarrow 1} \lambda(t) X_t = 0$ , therefore  $Y_1 = Z_1$ . ■

With this lemma at hand, establishing that the pair  $(H^*, \theta^*)$  given in the Theorem 3.1 is indeed an equilibrium is straightforward, as the following proposition demonstrates.

**Proposition 3.2** *Suppose that Assumptions 2.1, 2.2, 3.1 and 3.2 are satisfied. Then the pair  $(H^*, \theta^*)$ , where  $H^*(y, t)$  satisfies the partial differential equation (PDE) (3.7) with terminal condition (3.8), and the process  $\theta_t^*$  is given by (3.9), is an equilibrium.*

PROOF. Due to Proposition 3.1, the  $\theta_t^*$  defined by (3.9) is the optimal trading strategy given the admissible pricing rule  $H^*(y, t)$  which satisfies equation (3.7) and (3.8), if and only if: a)  $\theta_t^*$  is continuous with bounded variation, and b)  $Y_t^* = \theta_t^* + B_t^2$  satisfies  $Y_1^* = Z_1$ . Due to Lemma 3.1, we have that  $\theta_t^*$  is continuous with bounded variation, and  $Y_1^* = Z_1$ . Therefore  $\theta_t^*$  is an optimal trading strategy given the pricing rule  $H^*(y, t)$ .

On the other hand, due to the Lemma 3.1, for  $\theta^*$  given by (3.9),  $Y^*$  is a Brownian motion with  $Y_1^* = Z_1$ . Therefore, the rational pricing rule given  $\theta^*$  should be

$$H(y, t) = \mathbb{E} [f(y + B_1^2 - B_t^2)].$$

This pricing rule satisfies the PDE (3.7) with terminal condition (3.8). Therefore,  $H^*(y, t) = H(y, t)$  is a rational pricing rule. Hence, the pair  $(H^*, \theta^*)$  given in this proposition is an equilibrium. ■

To complete the proof of Theorem 3.1, we need to show uniqueness of the inconspicuous pricing rule in  $\mathcal{H}$ .

**Proposition 3.3** *The pricing rule  $H^*(y, t)$  which satisfies the PDE (3.7), with terminal condition (3.8), is the unique inconspicuous pricing rule.*

PROOF. From Proposition 3.3 and Lemma 3.1, it directly follows that  $H^*(y, t)$  satisfying the PDE (3.7) with terminal condition (3.8) is an inconspicuous equilibrium pricing rule. To prove uniqueness, consider some equilibrium inconspicuous pricing rule  $H$ . By definition, there exists a trading strategy  $\theta_t \in \mathcal{A}(H)$  such that the  $(H, \theta)$  is an equilibrium, and the total order process  $Y_t = \theta_t + B_t^2$  is a Brownian motion on  $\mathcal{F}_t^M$ . By the definition of equilibrium,

$$H(Y_t, t) = \mathbb{E} [f(Z_1) | \mathcal{F}_t^M] = \mathbb{E} [H(Y_1, 1) | \mathcal{F}_t^M].$$

Since  $Y_t$  is a Brownian motion on  $\mathcal{F}_t^M$ , and given the definition of admissible pricing rule,  $H$  must satisfy the PDE (3.7) with terminal condition  $H(y, 1) = h(y)$ , for some nondecreasing function  $h$  with  $\mathbb{E} [h^2(Y_1)] < \infty$ . Hence, to show uniqueness of  $H^*$  we need to demonstrate that  $h = f$  almost everywhere. Due to Proposition 3.1, it follows from the optimality of  $\theta$  that  $f(Z_1) = h(Y_1)$  and,

since  $\theta$  is inconspicuous,  $Y_1 \sim N(0, 1)$ . Since  $Z_1 \sim N(0, 1)$  by definition, one can have  $f(Z_1) = h(Y_1)$  if and only if  $f = h$  almost everywhere, hence  $H^*$  is indeed a unique inconspicuous pricing rule. ■

## 4 Non Markovian equilibrium

In this section I address the problem of existence of an equilibrium given by Definition 2.3 in the more general case of non Markovian pricing rule, i.e. I consider general weighting functions  $w(t)$  satisfying Definition 2.2 thus allowing the market maker to assign different weights to the information she receives.

As in the case of Markovian pricing rule, the existence of an equilibrium result is driven by the existence of a finite upper bound on the informed agent's value function, and the characterization of the trading strategies which attain it. This characterization is summarized in the following proposition.

**Proposition 4.1** *Suppose that Assumptions 2.1 and 2.2 are satisfied. Then, given an admissible pricing rule  $w(t)$  defined by (2.4) with  $\sigma_y^i < \sigma_y^{i+1}$  for any  $i$  and  $(H, w) \in \mathcal{H}$  with  $H$  satisfying the partial differential equation*

$$H_t(y, t) + \frac{w^2(t)}{2} H_{yy}(y, t) = 0 \quad (4.1)$$

*an admissible trading strategy  $\theta^* \in \mathcal{A}(H, w)$  is optimal for insider if and only if:*

1. *The process  $\theta_t^*$  is continuous and has bounded variation.*
2. *The weighted total order,  $\xi_t^* = \int_0^t w(s) d\theta_{s-}^* + \int_0^t w(s) dB_s^2$  satisfies*

$$h_i(\xi_{t_i}^*) = H(\xi_{t_i}^*, t_i) = F(Z_{t_i}, t_i). \quad (4.2)$$

Therefore, as in the case of Markovian pricing rule, the optimal strategy of the insider does not alter quadratic variation of total order process, does not add jumps to it and is uncorrelated with it. But, differently from the Markovian case, it follows from (4.2) that in this setting it is *optimal* for

the insider to reveal her information not only at the market terminal time, but also in the interim times whenever the market maker changes her weighting function.

PROOF.

(Sufficiency) As in the proof of Proposition 3.1, for any admissible trading strategy we have

$$\begin{aligned} \mathbb{E} [X_1^\theta] = & \mathbb{E} \left[ \int_0^1 (F(Z_s, s) - H(\xi_{s-}, s)) d\theta_s + \int_0^1 \theta_{s-} dF(Z_s, s) + \int_0^1 F_z(Z_s, s) d[\theta^c, Z]_s \right. \\ & \left. - \int_0^1 H_\xi(\xi_{s-}, s) w(s) d[\theta^c]_s - \int_0^1 H_\xi(\xi_{s-}, s) w(s) d[\theta^c, B^2]_s - \sum_{s \leq 1} \Delta H(\xi_s, s) \Delta \theta_s \right]. \end{aligned}$$

On the other hand, consider the functions

$$J^i(\xi, z) = \int_\xi^{\xi^*(z)} (F(z, t_i) - H(x, t_i)) dx,$$

where  $\xi_i^*(z)$  is the solution of  $H(\xi_i^*(z), t_i) = F(z, t_i)$ . For  $t \leq t_i$  let

$$V^i(\xi, z, t) = \mathbb{E} \left[ J^i \left( \xi + \int_t^{t_i} w(s) dB_s^2, z + \int_t^{t_i} \sigma_z(s) dB_s^1 \right) \right].$$

These functions are well defined (it is easy to check that  $\mathbb{E} [ |J^i(\int_0^{t_i} w(s) dB_s^2, Z_{t_i})| ] < \infty$ ) and satisfy the partial differential equation

$$V_t^i(\xi, z, t) + \frac{w^2(s)}{2} V_{\xi\xi}^i(\xi, z, t) + \frac{\sigma_z^2(t)}{2} V_{zz}^i(\xi, z, t) = 0$$

with terminal condition  $V^i(\xi, z, t_i) = J^i(\xi, z)$ . Therefore,  $V^i(\xi, z, t_i) \geq V(\xi_i^*(z), z, t_i) = 0$  for any fixed  $z$  and any  $\xi \neq \xi_i^*(z)$ . Moreover, since  $H(\xi, t)$  is a nondecreasing continuous function of  $\xi$ , and due to the definition of an admissible pricing rule, we have

$$V_\xi^i(\xi, z, t) + F(z, t) - H(\xi, t) = 0.$$

Define the function  $V$  as

$$V(\xi, z, t) = \sum_{i < n: t \leq t_i} \left( \frac{1}{\sigma_y^i} - \frac{1}{\sigma_y^{i+1}} \right) V^i(\xi, z, t) + \frac{1}{\sigma_y^n} V^n(\xi, z, t)$$

Notice that, due to the properties of the functions  $V^i$ , we have that  $V$  is well defined and satisfies the partial differential equation

$$V_t(\xi, z, t) + \frac{w^2(s)}{2} V_{\xi\xi}(\xi, z, t) + \frac{\sigma_z^2(t)}{2} V_{zz}(\xi, z, t) = 0, \quad (4.3)$$

with conditions  $V(\xi, z, t_i) = \left( \frac{1}{\sigma_y^i} - \frac{1}{\sigma_y^{i+1}} \right) J^i(\xi, z) + V(\xi, z, t_i+)$  if  $i < n$  and  $V(\xi, z, t_n) = \frac{1}{\sigma_y^n} J^n(\xi, z)$ . Moreover, we have

$$V_\xi(\xi, z, t) + \frac{F(z, t) - H(\xi, t)}{w(t)} = 0. \quad (4.4)$$

As a consequence, we can express  $\mathbb{E} [X_1^\theta]$  in terms of  $V$  as (notice that  $\int_0^t \int_0^u w(s) dB_s^2 dF(Z_u, u)$  is a martingale)

$$\begin{aligned} \mathbb{E} [X_1^\theta] &= \mathbb{E} \left[ - \int_0^1 V_\xi(\xi_{s-}, Z_s, s) w(s) d\theta_s - \int_0^1 V_z(\xi_{s-}, Z_s, s) dZ_s \right. \\ &\quad - \int_0^1 V_{z\xi}(\xi_{s-}, Z_s, s) w(s) d[\theta^c, Z]_s - \int_0^1 V_{\xi\xi}(\xi_{s-}, Z_s, s) w^2(s) d[\theta^c]_s \\ &\quad \left. - \int_0^1 V_{\xi\xi}(\xi_{s-}, Z_s, s) w^2(s) d[\theta^c, B^2]_s - \sum_{s \leq 1} \Delta(w(s) V_\xi(\xi_s, Z_s, s)) \Delta\theta_s \right]. \end{aligned}$$

On the other hand, by applying the Itô formula for semimartingales to  $V$  directly ([14], Theorem II.6.33, p. 81) and removing martingale terms we get

$$\begin{aligned} \mathbb{E} [X_1^\theta] &= \mathbb{E} \left[ V(0, Z_0, 0) - \sum_{i=1}^{n-1} \left( \frac{1}{\sigma_y^i} - \frac{1}{\sigma_y^{i+1}} \right) J^i(\xi_{t_i}, Z_{t_i}, t_i) - \frac{1}{\sigma_y^n} J^n(\xi_{t_i}, Z_{t_i}, t_i) \right. \\ &\quad \left. - \frac{1}{2} \int_0^1 V_{\xi\xi}(\xi_{s-}, Z_s, s) w^2(s) d[\theta^c]_s + \sum_{s \leq 1} [\Delta V(\xi_s, Z_s, s) - V_\xi(\xi_s, Z_s, s) \Delta\xi_s] \right]. \end{aligned}$$

Moreover, due to the properties of  $V$  we have

$$\sum_{s \leq 1} (\Delta V(\xi_s, Z_s, s) - V_\xi(\xi_s, Z_s, s) \Delta \xi_s) \leq 0, \quad (4.5)$$

$$- \int_0^1 \frac{V_{\xi\xi}(\xi_s, Z_s, s) w^2(s)}{2} d[\theta^c]_s \leq 0, \quad (4.6)$$

$$-J^i(t_i, \xi_{t_i}, Z_{t_i}) \leq 0. \quad (4.7)$$

The above inequalities become equalities if and only if the following conditions hold:  $\Delta\theta = 0$  for equation (4.5);  $[\theta^c]_1 = 0$  for equation (4.6);  $H(\xi_{t_i}^*, t_i) = F(Z_{t_i}, t_i)$  for equations (4.7).

Therefore, we have that

$$\mathbb{E} [X_1^\theta] \leq V(0, Z_0, 0).$$

This expression holds with equality if and only if  $\theta$  is continuous with bounded variation and condition (4.2) is satisfied. Hence, if  $\theta$  is such that these conditions are satisfied, then it is optimal.

(Necessity) Consider the process given by

$$X_t = G(Z_t, t) = \sum_{i=1}^n \mathbb{E} [h_i^{-1}(F(Z_{t_i}, t_i)) | \mathcal{F}_t^I] 1_{\{t \in (t_{i-1}, t_i]\}}$$

with  $X(0) = \mathbb{E} [h_1^{-1}(F(Z_{t_1}, t_1)) | \mathcal{F}_0^I]$  where  $h_i^{-1}$  is inverse of  $H(y, t_i)$ . This process is well defined since  $H$  is an admissible pricing rule.

Consider the trading strategy given by  $\theta_0 = 0$  and  $d\theta_t = \sum_{i=1}^n \frac{X_t - \xi_t}{\sigma_y^i(t_i - s)} 1_{\{t \in (t_{i-1}, t_i]\}} dt$ . In this case, we can solve the stochastic differential equation for  $\xi$  on each interval  $[t_{i-1}, t_i]$  to get

$$\xi_t = X_t - (t_i - t) \left( \frac{X_{t_{i-1}} - \xi_{t_{i-1}}}{t_i - t_{i-1}} + \int_{t_{i-1}}^t \frac{1}{t_i - s} dX_s - \int_{t_{i-1}}^t \frac{\sigma_y^i}{t_i - s} dB_s^2 \right).$$

Notice that  $\xi_t$  is finite almost surely, therefore  $\theta_t$  has bounded variation almost surely. More-

over,  $H(\xi_{t_i}, t_1) = F(Z_{t_i}, t_i)$  almost surely, hence this choice of  $\theta$  gives

$$\mathbb{E} \left[ X_1^\theta \right] = V(0, Z_0, 0).$$

Since, by the sufficiency proof, we have that for any  $\tilde{\theta}_t$  which is either not continuous or does not satisfy equation (3.10)

$$\mathbb{E} \left[ X_1^{\tilde{\theta}} \right] < V(0, Z_0, 0) = \mathbb{E} \left[ X_1^\theta \right],$$

we know that any such  $\tilde{\theta}_t$  is not optimal. ■

From this characterization follows the existence of equilibrium result, as the next theorem demonstrates.

**Theorem 4.1** *Suppose that  $\sigma_z(t)$  and  $\sigma$  are such that there exists a piecewise constant function  $g(t) = \sum_{i=1}^n \alpha_i 1_{\{t \in (t_{i-1}, t_i]\}}$  with  $0 = t_0 < \dots < t_n = 1$ ,  $0 < \alpha_i < \alpha_{i+1}$  for any  $i$  and  $\sum_{i=1}^n \alpha_i^2 = 1$ , satisfying the following conditions:*

$$\Sigma_z(t) + \sigma^2 - \int_0^t g^2(s) ds > 0 \text{ for all } t \in [0, 1] \setminus \{t_i\}_{i=0}^n, \quad (4.8)$$

$$\Sigma_z(t_i) + \sigma^2 - \int_0^{t_i} g^2(s) ds = 0 \text{ for all } t_i, \quad (4.9)$$

$$\int_{t_{i-1}}^t \frac{1}{(\Sigma_z(s) + \sigma^2 - \int_0^s g^2(u) du)^2} ds < \infty \text{ for all } t \in [t_{i-1}, t_i] \text{ and any } i \leq n, \quad (4.10)$$

$$\lim_{t \rightarrow t_i} \int_{t_{i-1}}^t \frac{1}{\Sigma_z(s) + \sigma^2 - \int_0^s g^2(u) du} ds = \infty. \quad (4.11)$$

Then there exists an equilibrium and it is given by the weighting function  $w^*(s) = g(s)$ , the pricing rule  $H^*(\xi, t) = \mathbb{E} \left[ f \left( \xi + \int_t^1 g(s) dB_s^2 \right) \right]$ , and the trading strategy  $\theta_t^*$  satisfying  $\theta_0^* = 0$  and

$$d\theta_t^* = 1_{\{t \in (0, t_1]\}} \frac{(Z_t - \alpha_1 Y_t) \alpha_1}{\Sigma_z(t) + \sigma^2 - \int_0^t g^2(s) ds} dt + \sum_{i=1}^{n-1} 1_{\{t \in (t_i, t_{i+1}]\}} \frac{(Z_t - Z_{t_i} - \alpha_{i+1} (Y_t - Y_{t_i})) \alpha_{i+1}}{\Sigma_z(t) + \sigma^2 - \int_0^t g^2(s) ds} dt.$$



This theorem implies that if there are times  $t_i$  such that  $\Sigma_z(t_i) + \sigma^2 - \int_0^{t_i} w^2(s)ds = 0$ , and the intensity of private information arrival is fast enough at these points (i.e. (4.10) is satisfied), then it is: *a)* rational for the market maker to change her weighting function at these points and *b)* it is optimal for the insider to reveal her information at these times.

Moreover, notice that this equilibrium exists even when assumptions 3.1 and 3.2 insuring existence of Markovian equilibrium are not satisfied. That is, even if  $\Sigma_z(s) + \sigma^2 - s < 0$  for some  $s$ , there is a non Markovian equilibrium as long as there exists a piecewise linear increasing function  $g$ , the integral of which is bounding the realized variance of the insider signal  $(\Sigma_z(t) - \sigma^2)$  from below and satisfies the conditions of the theorem.

The proof of this theorem relies on linear filtering and deterministic time change.

PROOF. To demonstrate that  $((H^*, w^*), \theta^*)$  is an equilibrium, it is enough to show that

$$Y_t^* - Y_{t_i}^* \text{ is a Brownian motion on } [t_i, t_{i+1}] \text{ in its own filtration,} \quad (4.12)$$

where  $Y_t^* = \theta_t^* + B_t^2$  and

$$\alpha_{i+1} (Y_{t_{i+1}} - Y_{t_i}) = Z_{t_{i+1}} - Z_{t_i}. \quad (4.13)$$

Indeed, if these two conditions are satisfied, since  $\Sigma_z(t_i) + \sigma^2 - \int_0^{t_i} g^2(s)ds = 0$ , we will have that  $H^*(\xi_{t_i}, t_i) = F(Z_{t_i}, t_i)$ , therefore  $H^*$  is an admissible and rational pricing rule. Moreover, if condition (4.12) is satisfied, then  $\theta^*$  is continuous with bounded variation. Therefore it follows from Proposition 4.1 that  $\theta^*$  is optimal if condition (4.13) holds (notice that  $H^*$  satisfies PDE (4.1)).

Thus, to show that  $Y^*$  satisfies (4.12) and (4.13) is the next goal. The proof is by induction.

I) Consider the interval  $[0, t_1]$ . At  $t = 0$  we have  $Y_0 = 0$ ,  $Z_0 = v$  and  $Y_t$  satisfies the following stochastic differential equation on  $[0, t_1]$ :

$$dY_t = \frac{(Z_t - \alpha_1 Y_t) \alpha_1}{\Sigma_z(t) + \sigma^2 - \alpha_1^2 t} dt + dB_t^2$$

with  $dZ_t = \sigma_z(t)dB_t^1$ . From Theorem 10.3 of [12] (note that due to (4.10), the conditions of

the theorem are satisfied), we have that on the filtration  $(\mathcal{F}_t^Y)_{t < t_1}$  the stochastic differential equation for  $Y$  is

$$dY_s = \frac{(m_s - \alpha_1 Y_s) \alpha_1}{\Sigma_z(t) + \sigma^2 - \alpha_1^2 t} ds + dB_s^Y,$$

with

$$dm_s = \frac{\gamma_s \alpha_1}{\Sigma_z(t) + \sigma^2 - \alpha_1^2 t} dB_s^Y,$$

where  $B_t^Y$  is Brownian motion on  $\mathcal{F}_t^Y$ , and  $\gamma_s$  satisfies the following ODE

$$\dot{\gamma}_s = \sigma_z^2(s) - \frac{\gamma_s^2 \alpha_1^2}{(\Sigma_z(t) + \sigma^2 - \alpha_1^2 t)^2}$$

with initial condition  $\gamma_0 = \sigma^2$ .

Notice that  $\gamma_s = \Sigma_z(t) + \sigma^2 - \alpha_1^2 t$  is the unique solution of this ODE and initial condition.

Therefore on  $(\mathcal{F}_t^Y)_{t \leq t_1}$ , the process  $Y$  satisfies

$$dY_s = \frac{(B_s^Y - Y_s) \alpha_1^2}{\Sigma_z(t) + \sigma^2 - \alpha_1^2 t} ds + dB_s^Y.$$

The unique strong solution of this stochastic differential equation on  $[0, t_1)$  is  $Y_s = B_s^Y$  (see [10], Example 5.2.4). Hence, on the interval  $[0, t_1)$ , the process  $Y$  is a Brownian motion on its own (completed) filtration. By continuity of  $Y$ , this process is a Brownian motion on  $[0, t_1]$ .

To prove that  $\alpha_1 Y_{t_1} = Z_{t_1}$ , notice that

$$\alpha_1 Y_t^* = Z_t + \lambda(t) \left( -v + \int_0^t \frac{\alpha_1}{\lambda(s)} dB_s^2 - \int_0^t \frac{\sigma_z(s)}{\lambda(s)} dB_s^1 \right)$$

where  $\lambda(t) = \exp \left\{ - \int_0^t \frac{\alpha_1}{\Sigma_z(t) + \sigma^2 - \alpha_1^2 t} ds \right\}$ .

Note that a random variable  $\int_0^t \frac{\alpha_1}{\lambda(s)} dB_s^2 - \int_0^t \frac{\sigma_z(s)}{\lambda(s)} dB_s^1$  is normally distributed with mean 0 and variance  $\int_0^t \frac{\alpha_1^2 + \sigma_z^2(s)}{\lambda^2(s)} ds$ . Therefore, due to condition (4.9), if  $\lim_{t \rightarrow t_1} \int_0^t \frac{\alpha_1^2 + \sigma_z^2(s)}{\lambda^2(s)} ds < \infty$ , then  $\xi_{t_1} = \alpha_1 Y_{t_1} = Z_{t_1}$ .

On the other hand, if  $\lim_{t \rightarrow t_1} \int_0^t \frac{\alpha_1^2 + \sigma_z^2(s)}{\lambda^2(s)} ds = \infty$ , consider the process

$$X_t = \int_0^t \frac{\alpha_1}{\lambda(s)} dB_s^2 - \int_0^t \frac{\sigma_z(s)}{\lambda(s)} dB_s^1,$$

and a change of time  $\tau(t)$  given by

$$\int_0^{\tau(t)} \frac{\alpha_1^2 + \sigma_z^2(s)}{\lambda^2(s)} ds = t.$$

Then,  $W_s = X_{\tau(s)}$  is a Brownian motion. Hence, we can use the law of iterated logarithm to get

$$\begin{aligned} \limsup_{s \rightarrow \infty} \frac{W_s}{\sqrt{2s \log \log s}} &= 1 \\ \liminf_{s \rightarrow \infty} \frac{W_s}{\sqrt{2s \log \log s}} &= -1 \end{aligned}$$

or, in the original time,

$$\begin{aligned} \limsup_{t \rightarrow t_1} \frac{X_t}{\sqrt{2\Xi(t) \log \log(\Xi(t))}} &= 1 \\ \liminf_{t \rightarrow t_1} \frac{X_t}{\sqrt{2\Xi(t) \log \log(\Xi(t))}} &= -1 \end{aligned}$$

where  $\Xi(t) = \int_0^t \frac{\alpha_1^2 + \sigma_z^2(s)}{\lambda^2(s)} ds$ . Due to the conditions (4.8)-(4.11) in this case we have

$$\lim_{t \rightarrow t_1} \lambda^2(t) \Xi(t) \log \log(\Xi(t)) = 0,$$

therefore it follows that  $\lim_{t \rightarrow t_1} \lambda(t) X_t = 0$ , thus  $\xi_{t_1} = \alpha_1 Y_{t_1} = Z_{t_1}$ .

II) Suppose  $\alpha_j(Y_{t_j} - Y_{t_{j-1}}) = Z_{t_j} - Z_{t_{j-1}}$  for any  $j \leq i$ . Consider the interval  $[t_i, t_{i+1}]$ . At  $t = t_i$  we have  $\xi_{t_i} = Z_{t_i}$ , and  $\tilde{Y}_t = Y_t - Y_{t_i}$  satisfies the following stochastic differential equation on  $[t_i, t_{i+1}]$ :

$$d\tilde{Y}_t = \frac{\left(\tilde{Z}_t - \alpha_{i+1} \tilde{Y}_t\right) \alpha_{i+1}}{\Sigma_z(t) - \Sigma_z(t_i) - \alpha_{i+1}^2(t - t_i)} dt + dB_t^2$$

with  $\tilde{Z}_t = Z_t - Z_{t_i}$ , thus  $d\tilde{Z}_t = \sigma_z(t)dB_t^1$  and  $\tilde{Z}_{t_i} = 0$ . From Theorem 10.3 of [12] (note that, due to (4.10), the conditions of the theorem are satisfied), we have that on the filtration  $(\mathcal{F}_t^Y)_{t \in [t_i, t_{i+1})}$  the stochastic differential equation for  $Y$  is

$$d\tilde{Y}_s = \frac{(m_s - \alpha_{i+1}\tilde{Y}_s)\alpha_{i+1}}{\Sigma_z(t) - \Sigma_z(t_i) - \alpha_{i+1}^2(t - t_i)}ds + dB_s^Y,$$

with

$$dm_s = \frac{\gamma_s \alpha_{i+1}}{\Sigma_z(t) - \Sigma_z(t_i) - \alpha_{i+1}^2(t - t_i)}dB_s^Y,$$

where  $B_t^Y$  is Brownian motion on  $\mathcal{F}_t^Y$ , and  $\gamma_s$  satisfies the following ODE

$$\dot{\gamma}_s = \sigma_z^2(s) - \frac{\gamma_s^2 \alpha_{i+1}^2}{(\Sigma_z(t) - \Sigma_z(t_i) - \alpha_{i+1}^2(t - t_i))^2}$$

with initial condition  $\gamma_{t_i} = 0$ .

Notice that  $\gamma_s = \Sigma_z(t) - \Sigma_z(t_i) - \alpha_{i+1}^2(t - t_i)$  is the unique solution of this ODE and initial condition. Therefore on  $(\mathcal{F}_t^Y)_{t \in [t_i, t_{i+1})}$ , the process  $Y$  satisfies

$$dY_s = \frac{(B_s^Y - Y_s)\alpha_{i+1}^2}{\Sigma_z(t) - \Sigma_z(t_i) - \alpha_{i+1}^2(t - t_i)}ds + dB_s^Y.$$

The unique strong solution of this stochastic differential equation on  $[t_i, t_{i+1})$  is  $Y_s = B_s^Y$  (see [10], Example 5.2.4). Hence, on the interval  $[t_i, t_{i+1})$ , the process  $Y$  is a Brownian motion on its own (completed) filtration. By continuity of  $Y$ , this process is a Brownian motion on  $[t_i, t_{i+1}]$ . To prove that  $\alpha_{i+1}\tilde{Y}_{t_{i+1}} = \tilde{Z}_{t_{i+1}}$ , notice that

$$\alpha_{i+1}Y_t^* = Z_t + \lambda(t) \left( \int_{t_i}^t \frac{\alpha_{i+1}}{\lambda(s)} dB_s^2 - \int_{t_i}^t \frac{\sigma_z(s)}{\lambda(s)} dB_s^1 \right)$$

where  $\lambda(t) = \exp \left\{ - \int_{t_i}^t \frac{\alpha_{i+1}}{\Sigma_z(t) - \Sigma_z(t_i) - \alpha_{i+1}^2(t - t_i)} ds \right\}$ .

Therefore, due to conditions (4.8)-(4.11) we have, exactly as in the previous case,  $\alpha_{i+1}\tilde{Y}_{t_{i+1}} = \tilde{Z}_{t_{i+1}}$ .

By the principle of mathematical induction, conditions (4.12) and (4.13) hold for any  $i$ . ■

## 5 Conclusion

This paper demonstrates that, in the presence of *dynamic* private information of the insider, and under minimal restrictions on the admissible trading strategies, an equilibrium exists and there is a unique Markovian pricing rule (as a function of total order process) that admits inconspicuous equilibrium. Moreover, the optimal insider trading strategy is based on the market estimates of the fundamentals, rather than on the stock price: the insider buys the stock when the market overestimates the fundamental value, and sells it otherwise, thus leading to higher informativeness of the stock price. Furthermore, this induces convergence of the price to the fundamental value at the terminal time in the case of Markovian pricing rule and at some some set of times (which include the terminal time) in the case of non Markovian pricing rule.

Future research can be conducted along the following directions: assumptions on the market parameters could be further relaxed, and a more general model of the total order of the noisy traders could be considered. The model can also be generalized further by allowing for potential bankruptcy of the firm issuing the stock, with the time of bankruptcy defined as the random time at which the underlying process governing the firm value hits a given barrier.

## A Proof of Remark 3.1

Suppose Assumption 2.2 is satisfied,  $\lim_{t \rightarrow 1} \Xi(t) = \infty$  and conditions (3.3) and (3.4) hold. Then L'Hôpital rule will give (notice that due to (3.4) and continuity of  $\sigma_z(t)$  in the vicinity of 1 we have  $\lim_{t \rightarrow 1} (1 + \sigma_z^2(t)) < 2$ )

$$\lim_{t \rightarrow 1} \lambda^2(t) \Xi(t) \log \log (\Xi(t)) = \frac{1}{2} \lim_{t \rightarrow 1} (1 + \sigma_z^2(t)) \lim_{t \rightarrow 1} (\Sigma_z(t) + \sigma^2 - t) \log \log (\Xi(t)).$$

Since by L'Hôpital rule we have

$$\lim_{t \rightarrow 1} \lambda^2(t) \Xi(t) = 0,$$

it follows that

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow 1} \lambda^2(t) \Xi(t) \log \log (\Xi(t)) \leq \frac{1}{2} \lim_{t \rightarrow 1} (1 + \sigma_z^2(t)) \lim_{t \rightarrow 1} (\Sigma_z(t) + \sigma^2 - t) \log \log (\lambda^{-2}(t)) \\ &= \frac{1}{2} \lim_{t \rightarrow 1} (1 + \sigma_z^2(t)) \lim_{t \rightarrow 1} (\Sigma_z(t) + \sigma^2 - t) \log \left( \int_0^t \frac{1}{\Sigma_z(s) + \sigma^2 - s} ds \right) \\ &= \frac{1}{2} \lim_{t \rightarrow 1} (1 + \sigma_z^2(t)) \lim_{t \rightarrow 1} \frac{\log(f(t))}{f'(t)}, \end{aligned}$$

where  $f(t) = \int_0^t \frac{1}{\Sigma_z(s) + \sigma^2 - s} ds$  and  $\lim_{t \rightarrow 1} f(t) = \infty$ . Since  $\lim_{x \rightarrow \infty} \frac{\log(x)}{x^\alpha} = 0$ , for any  $\alpha > 0$  we need to show that

$$\limsup_{t \rightarrow 1} \frac{f^\alpha(t)}{f'(t)} < \infty \tag{A.14}$$

for some  $\alpha > 0$  to establish (3.6).

Consider any  $\alpha \in (0, 1)$  and denote by

$$0 < g(t) = \frac{f^\alpha(t)}{f'(t)},$$

then for  $t \geq t^*$  we have

$$f^{1-\alpha}(t) = (1 - \alpha) \int_{t^*}^t \frac{1}{g(s)} ds + c$$

where  $c$  is some positive constant. Due to this expression and since  $\lim_{t \rightarrow 1} f(t) = \infty$ ,  $\alpha < 1$  and,

due to (3.1) ,  $f(t) < \infty$  for any  $t \in [0, 1)$  we must have

$$\lim_{t \rightarrow 1} g(t) = 0.$$

Thus (A.14) holds and therefore (3.6) is established.

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